APPLICATION OF THE MULTIGRID METHOD TO GRID GENERATION

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ABSTRACT

The multigrid method (MGM) has been used to numerically solve the pair of nonlinear elliptic equations commonly used to generate two-dimensional boundary-fitted coordinate systems. Two different geometries are considered: one involving a coordinate system fitted about a circle and the other selected for an impinging jet flow problem. MGM uses a nest of grids from finest (upon which the solution is sought) to coarsest and is based on the idea of using relaxation sweeps to smooth the error (equivalent to eliminating high frequency Fourier components of the error). Thus most of the computational work is done on coarser subgrids to eliminate longer wave length components of the error. Two different relaxation schemes are tried: one is successive point overrelaxation and the other is a four-color scheme vectorizeable to take advantage of a parallel processor computer for greater computational speed. Results using MGM are compared with those using SOR (doing successive overrelaxations with the corresponding relaxation scheme on the fine grid only). It is found that MGM becomes significantly more effective than SOR as more accuracy is demanded and as more corrective grids, or more grid points, are used. For the accuracy required here, it is found that MGM is two to three times faster than SOR in computing time. With the four-color relaxation scheme as applied to the impinging jet problem the advantage of MGM over SOR is not as great. Perhaps this is due to the effect of a poor initial guess on MGM for this problem.

The multigrid method (MGM) [1] can numerically solve linear or nonlinear elliptic partial differential equations more rapidly than conventional means of solution such as successive overrelaxation (SOR). MGM can be applied to the numerical solution of partial differential equations not amenable to numerical solution by fast direct matrix solvers such as diagonal decomposition. Thus it was deemed desirable to apply MGM to the numerical solution of the system of nonlinear elliptic equations commonly used to generate boundary-fitted coordinate systems, especially when the number of grid points is large. The standard elliptic equations for a typical mapping, shown schematically in Figure 1, are

$$L_1(x,y) = \alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} + J^2(Px_\xi + Qx_\eta) = 0 \tag{1}$$

$$L_2(x,y) = \alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} + J^2(Py_\xi + Qy_\eta) = 0 \tag{2}$$

where

$$\begin{aligned} \alpha &= x_\eta^2 + y_\eta^2 & \beta &= x_\xi x_\eta + y_\xi y_\eta \\ \gamma &= x_\xi^2 + y_\xi^2 & J &= x_\xi y_\eta - x_\eta y_\xi \end{aligned} \tag{3}$$

and P and Q are functions of ξ and η. Dirichlet conditions are specified on all boundaries of the computational space including the interior slit (which maps to the body in the physical space). Each side of the slit has a set of Dirichlet data with a common value for each of the endpoints of the slit.

The basic idea of MGM is to do most of the computational work on coarser corrective grids containing far fewer points than the finest grid upon which the solution is sought. The grids form a nest, each coarser grid having twice the mesh spacing in each coordinate direction of the previous finer grid. In Figure 2 which represents the Full Approximation Storage scheme of [1]: $u = (x,y)$, $L = \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix}$ such that Eqs. (1) and (2) become $Lu = F = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, $1 \leq k \leq M$ (k representing the k^{th} grid with M the finest), $\phi = (x,y)$ on the boundaries of the computational space (Dirichlet values so that Λ is an

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[1] A. Brandt, Math. of Comp., Vol. 31, No. 138, April 1977, pp. 333-390.

identity operator) and superscripts refer to discretized quantities on the k^{th} grid. (All operations involving ϕ in the flow chart can be ignored, since the Dirichlet conditions are constant on all the grids.) The main idea behind MGM is that relaxation sweeps are a smoothing process which eliminate the highest frequency Fourier components of the error on any grid. First, starting with an initial guess for the solution, several sweeps are carried out on the finest grid to eliminate high frequency components of the error. The smoothed out error is represented by the residual $f^M = L^M U^M - L^M u^M$ and the correction $U^M - u^M$ (where U^M is the exact discrete solution on the finest M^{th} grid). The residual, consisting mainly of longer wave-length Fourier components, is dealt with by solving its coarser-grid approximation

$$L^{M-1} U^{M-1} - L^{M-1} I_M^{M-1} u^M = I_M^{M-1} f^M \qquad (4)$$

for U^{M-1}, which is represented by $\bar{F}^k$ for $k = M-1$ in the lower right box of Figure 2. The symbol I_k^{k-1} means interpolation of a quantity from the k^{th} grid to the (k-1)st grid. Eq. (4) is solved in the same way as the original equation on the finest grid. If solution of (4) is obtained after several relaxation sweeps, the coarse grid approximation $U^{M-1} - I_M^{M-1} u^M$ to the smoothed out function $U^M - u^M$ is added to u^M. That is $u^M \leftarrow u^M + I_{M-1}^M (u^{M-1} - I_M^{M-1} u^M)$, which is the expression in the lower left box for $k = M$. The new u^M is a better approximation to the solution U^M and is the starting point for more relaxation sweeps for the original set of Eqs. (1) and (2) on the finest grid. If convergence is obtained, the process is complete; if not, the process returns to the coarser grid to sweep the residual equation again. If it doesn't converge after a few sweeps, then the next coarser grid is used to eliminate long wave length errors for the residual equation, etc. Each residual equation has a corresponding residual equation and correction on the next coarser grid. (The residuals were weighted locally as in [1].)

Figure 3 shows computer drawn body-fitted coordinate systems generated to a specific accuracy using MGM and SOR (the two coordinate systems coincide). The relaxation scheme used was successive point overrelaxation. According to the notation used in Figure 1, m and n are 81 and 21, respectively; the slit end points are (ξ_{33}, η_{13}) and (ξ_{53}, η_{13}), respectively; $(x_\ell, y_b) = (-8.4, -8.0)$

and $(x_r, y_t) = (7.6, 0.0)$; $\Delta x = .2$ and $\Delta y = .4$; $\Delta\xi = \Delta\eta = 1$; and the body is a circle of radius one centered at $(x,y) = (0, -3.2)$. P and Q were set to zero in Eqs. (1) and (2). An experimentally determined, essentially optimum overrelaxation factor of 1.7 was used in the successive point overrelaxation sweeps in both MGM and the SOR method. All coarser corrective grids contain grid points on the slit. The initial guess for $x(\xi,\eta)$, $y(\xi,\eta)$ in the computational space is obtained by extending the Dirichlet data at the outer boundaries throughout the space except at the slit, where the body Dirichlet data are used. The convergence criterion for the solution of Eqs. (1) and (2) was that both L_2-error norms (one for each equation) be less than an input value $||E||_{L_2}$. (This will be called satisfaction of $||E||_{L_2}$.) For Figure 3, $||E||_{L_2} = .001$. To satisfy this criterion, MGM used 32.5 WU and 16.08 CP seconds compared to 66.0 WU and 22.17 CP seconds for SOR. (A work unit (WU) is the equivalent of one SOR sweep on the finest grid, and CP seconds refer to central processor seconds used on the Texas Instruments Advanced Scientific Computer (TI-ASC).) For $||E||_{L_2} = .01$, MGM used 20 WU compared to 29 WU for SOR; CP time was the same for both methods (due mainly to the additional computational work in computing residuals in MGM). The results show that the effectiveness of MGM increases (compared to SOR) as the error norm decreases. This is consistent with the fact that the remaining longer wave length errors are eliminated more slowly using SOR. The parameters $\delta = .3$, $\zeta = .3$ were used to control the flow of MGM. The parameter δ determines the convergence test on each grid and the parameter ζ determines how fast the convergence must be (how fast the high frequency components are eliminated) on each grid. Whenever $\zeta < (||E||^k_{L_2})^{i+1}/(||E||^k_{L_2})^i$ on a k^{th} grid, MGM will then process on the coarser (k-1)st grid with an error norm to be satisfied equal to $\delta(||E||^k_{L_2})^{i+1}$. (Superscripts i,k refer to the i^{th} relaxation sweep and the k^{th} grid, respectively.) These parameters are used as in [1], have a range $(0 < \delta < 1;\ 0 < \zeta < 1)$, and greatly influence the performance of MGM. The present choice is not necessarily optimum but was the best of a number of choices tried in the unit square.

Figure 4 shows a computer drawn body-fitted coordinate system, similar to Figure 3, generated with MGM and satisfying $||E||_{L_2} = .001$. The grid parameters are (see Figure 1): m and n equal to 129 and 81, respectively;

slit end points of (ξ_{49}, η_{49}) and (ξ_{81}, η_{49}), respectively; (x_ℓ, y_b) = (-7.68, -8.0) and (x_r, y_t) = (7.68, 0.0); Δx = .12 and Δy = .1; $\Delta\xi = \Delta\eta = 1$; and the circle of radius one was centered again at (0, -3.2). To satisfy $||E||_{L_2}$ = .001 MGM used 21.863 WU and 70.67 CP seconds compared to 102.0 WU and 217.83 CP seconds used by SOR. This represents a significant saving of computer time by MGM. To satisfy $||E||_{L_2}$ = .01 MGM used 10.863 WU compared to 17.0 WU used by SOR with CP time essentially the same. These results, along with those for Figure 3, show that MGM is more effective, compared to SOR, when more corrective grids are used and more accuracy is required. Figure 4 has five corrective grids and Figure 3 has three corrective grids (including the finest). The parameters δ = .03 and ζ = .2 controlled MGM for Figure 4. Choosing smaller δ and ζ makes it more likely that all the coarser corrective grids will be used, which is desirable.

Figure 5 shows a computer drawn body-fitted coordinate system generated using MGM and satisfying $||E||_{L_2}$ = .001. SOR was also used to generate this grid and is in excellent agreement with MGM. The geometry is motivated by an impinging jet flow problem that is planned to be run on this grid. The flow from the channel interacts with the solid body on the right. The computational space has the same shape as the physical space except that the body is replaced by a slit. Excluding the channel, the grid consists of 137 points in the horizontal direction by 97 points in the vertical direction. The grid for the channel itself consists of 25 horizontal grid points by 33 vertical grid points. The slit (and body) are 49 grid points long. Corner points on the body and channel have been excluded from the grid. Exponential grid spacing was used along various parts of the horizontal and vertical boundaries of the grid. In an attempt to preserve this boundary spacing in the grid interior non-zero P and Q were used. Although grid lines are still bent near the boundaries, they are not bent as much as when P = Q = 0 was tried. To compute this grid (which had 4 corrective grids, including the finest) MGM was "vectorized" on the TI-ASC since it is a parallel processor machine. To accomplish vectorization, which cut computing time by a factor of six, a four-color relaxation scheme was used (i.e., even points of even rows were relaxed simultaneously; odd points of even rows; etc.). With this scheme

MGM used 82.781 WU and 45.57 CP seconds to satisfy $||E||_{L_2} = .001$ when using an overrelaxation factor (RF) of 1.8 on the finest grid and relaxation factors of 1.6, 1.4, and 1.2 for the progressively coarser grids. (Varying RF in this way improved MGM's performance.) SOR (with the four-color scheme) used 170.0 WU and 76.68 CP seconds using a relaxation factor of 1.8, which is about optimum for this SOR. MGM used 60.641 WU and 36.67 CP seconds to satisfy $||E||_{L_2} = .001$ when RF's of 1.6, 1.4, 1.2, 1.0 were used on progressively coarser grids (with 1.6 used for the finest grid). With these RF's MGM used 26.016 WU to satisfy $||E||_{L_2} = .01$ compared to 82.0 WU used by SOR with RF = 1.8. The parameters $\delta = .05$, $\eta = .95$ were used for MGM which was divergent for $\eta < .9$. MGM should perform better with a better initial guess than used here. (The horizontal straight lines in the initial guess were discontinuous at the right-most boundary.)

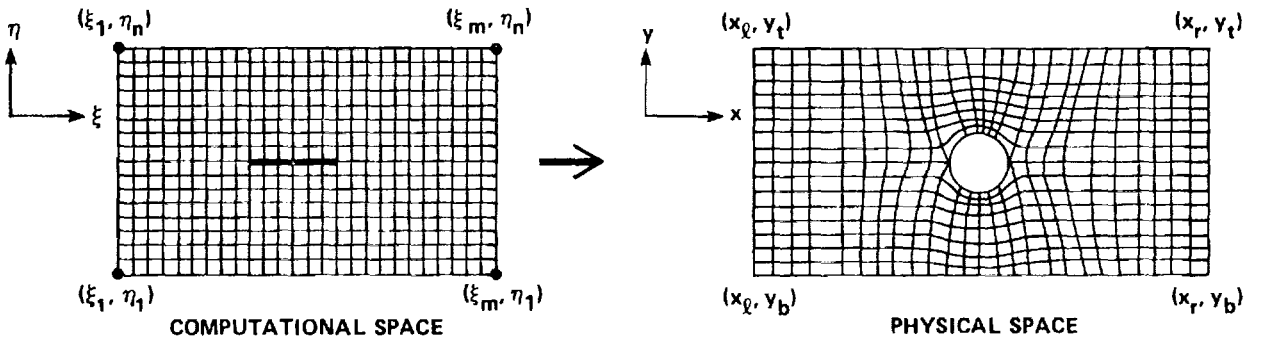

Figure 1

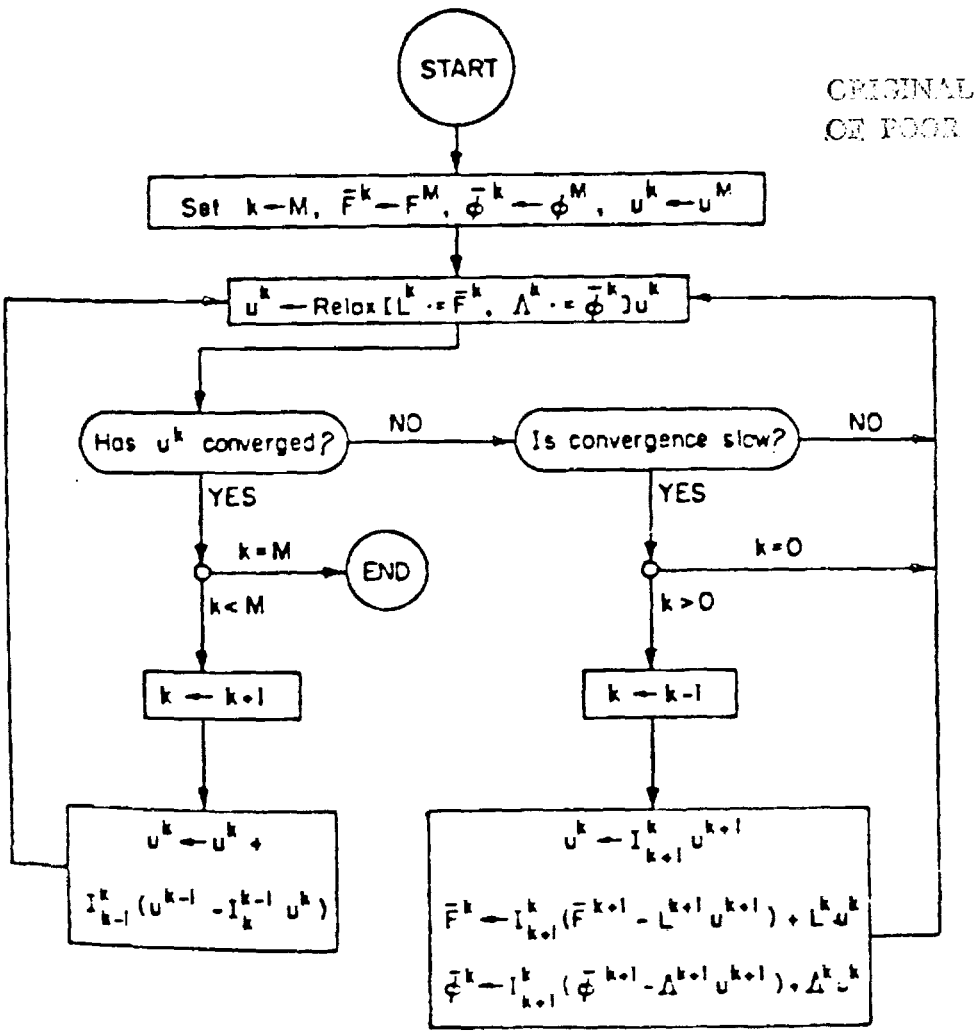

Figure 2

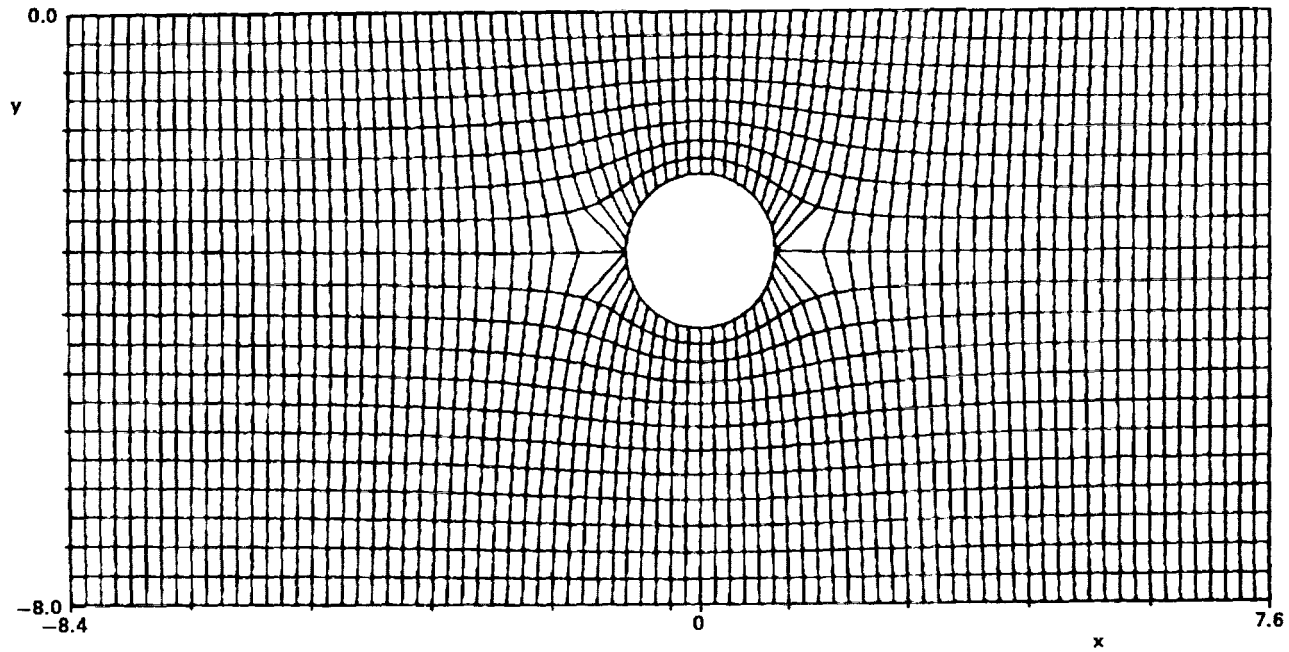

Figure 3

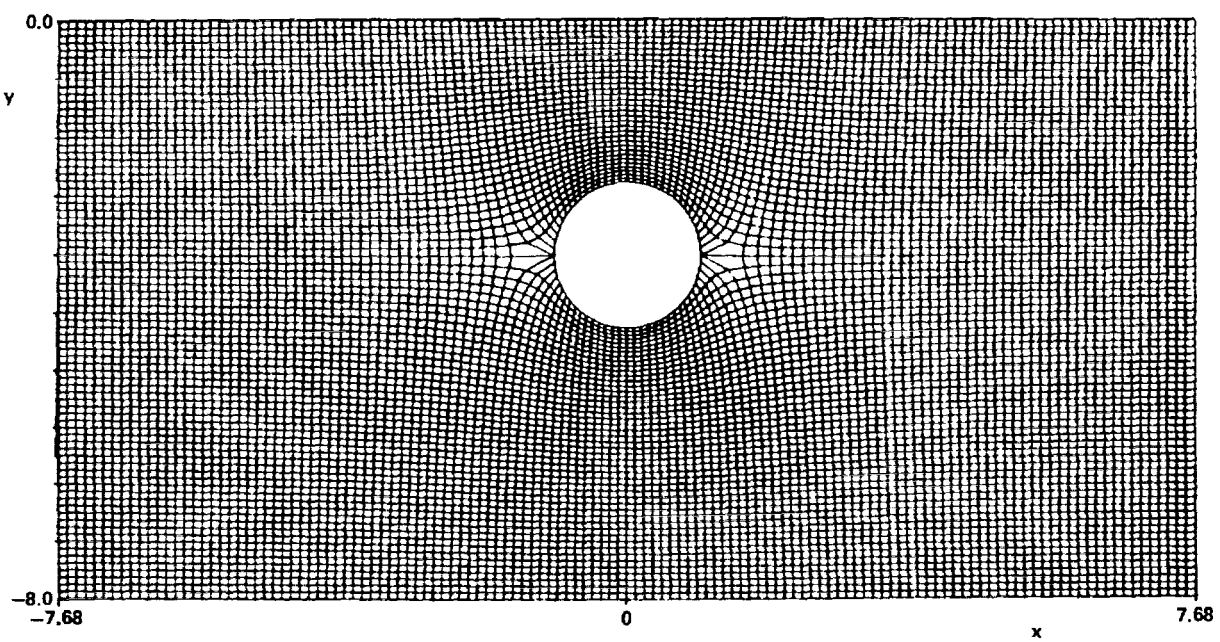

Figure 4

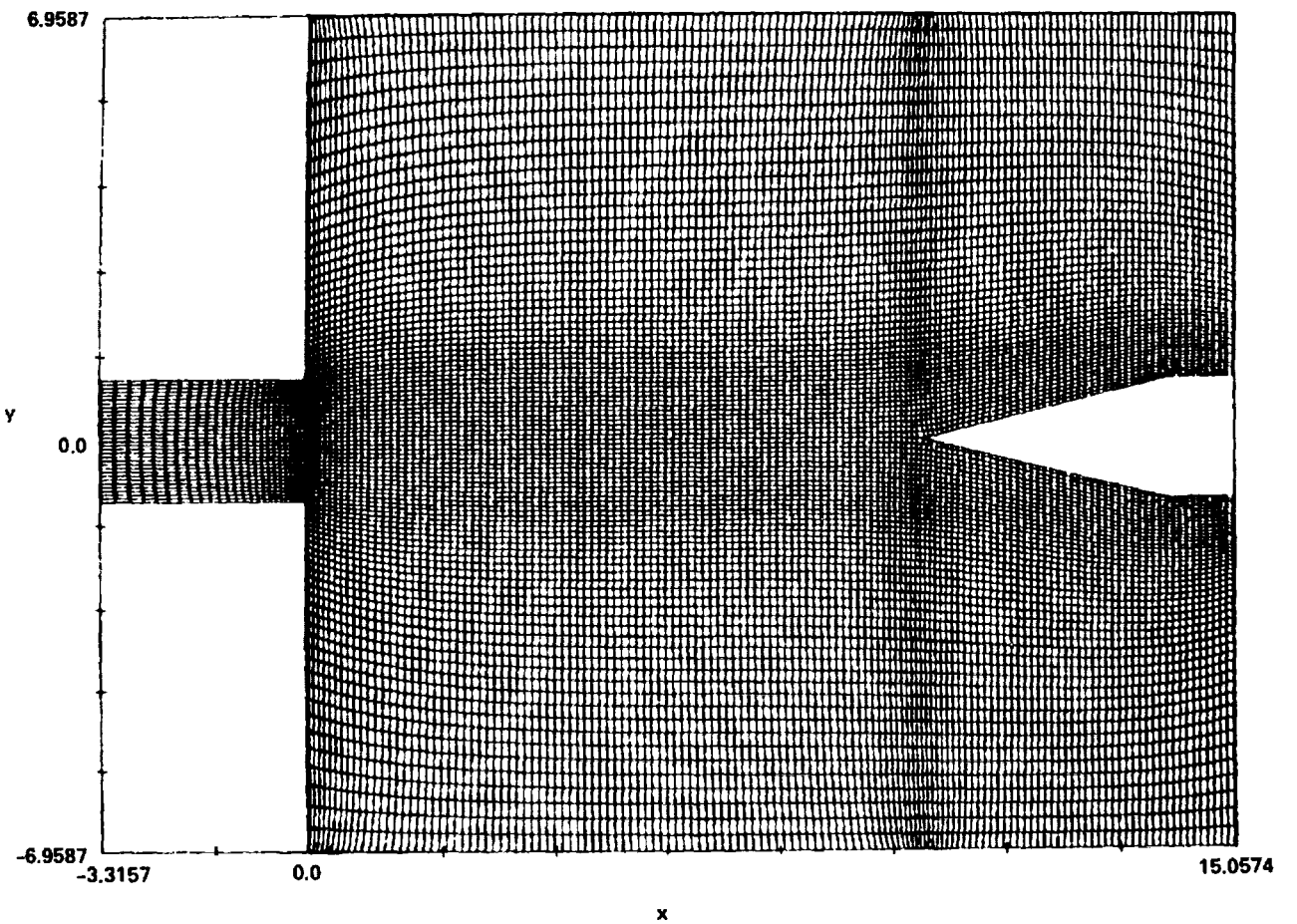

Figure 5